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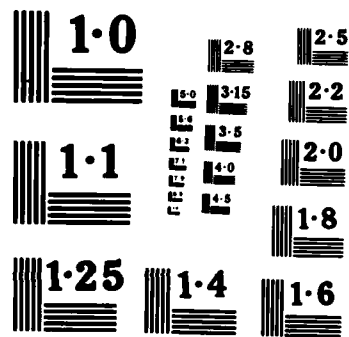
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**The Pennsylvania State University**  
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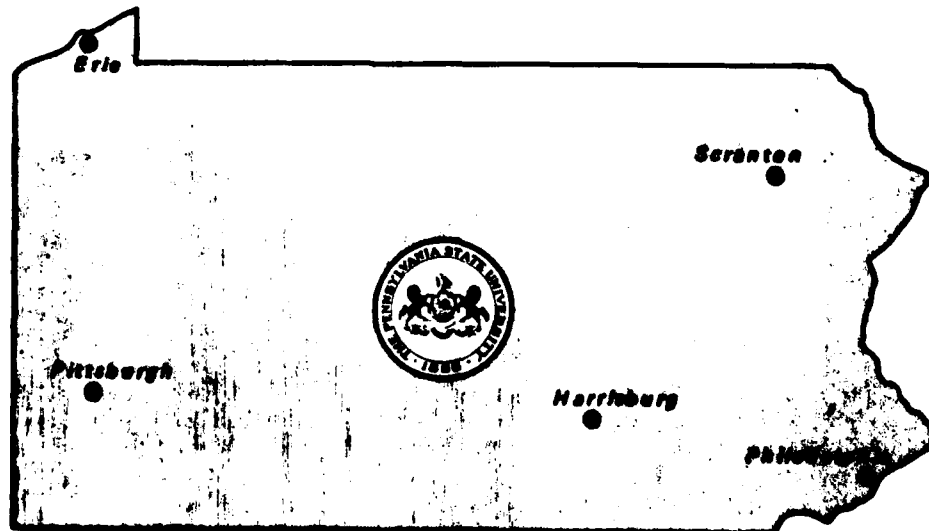
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Number 56: August 1985

TESTS FOR PATTERNED ALTERNATIVES

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Professor of Statistics  
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# Abstract

This paper treats the problem of testing for a patterned alternative in a one- or two-way layout. Ordered and umbrella alternatives are special cases. Statistics based on combined rankings and on pairwise rankings are developed. Modifications necessary to incorporate covariates are also included.

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Key words: Rank test, ordered alternative, umbrella alternative.



## 1. Introduction

In this paper we consider the problem of testing for a patterned alternative. In two sample comparisons it is well known that, when appropriate, a one sided critical region is more powerful than a two sided region. The situation with more than two samples is more complex since there are various ways to specify an alternative. These alternatives are the patterns we wish to detect.

The ordered alternative in a one-way layout is a generalization of the one sided alternative. There is an extensive literature in this area and it continues to be an active area of research. See Barlow et. al. (1972) for an overview. A monotonic ordering of locations is only one of many patterns that may be of interest to the researcher. Umbrella alternatives, studied by Mack and Wolfe (1981), provide another important example.

In Section 2 we consider testing for patterned alternatives in a one-way layout. Given a pattern, we show how to construct a most efficient rank test within a certain class of rank tests. We provide two versions of these test statistics: one based on average ranks after ranking the combined data and one based on pairwise rank statistics formed by ranking only within each of the pairs of samples. The statistics based on the average ranks have the advantage of

being quite easy to apply in a large variety of situations. Some researchers may prefer the statistic based on pairwise comparisons because the pairwise statistics provide direct information on the sources of statistical significance.

We also discuss the problem of detecting when a sample pattern conforms to a pattern within a finite set of possible alternative patterns. In Section 4 we illustrate this approach on the umbrella alternative when the peak is unknown. Tables of critical values are provided for this problem.

In Section 3 we consider the problem of testing for a pattern when there are covariate measurements available.

## 2. The Test Statistics

Suppose  $X_{ij}$  is distributed with continuous cumulative distribution function  $F(x-\theta_j)$  for  $j=1,\dots,k$  and  $i=1,\dots,n_j$ , and the observations are mutually independent. We wish to consider tests of  $H_0: \theta_1=\dots=\theta_k$  versus  $H_A: \theta_j = \theta_0 + \theta c_j$ ,  $\theta > 0$ ,  $j=1,\dots,k$ , where  $c_1,\dots,c_k$  is a given set of constants. The set of constants  $c_1,\dots,c_k$  specifies the pattern to be detected. For example,  $c_1 < \dots < c_k$  determines an ordered alternative and  $c_1 < \dots < c_t > c_{t+1} > \dots > c_k$  determines an umbrella alternative with peak at  $\theta_t$ . The researcher will generally specify the constants; and, unless there are

indications to the contrary, we recommend equally spaced constants.

Let  $R_{ij}$  be the rank of  $X_{ij}$  in the combined data and let  $\bar{R}_j = n_j^{-1} \sum_i R_{ij}$  be the average rank of the  $j$ th group. We will consider statistics of the form  $V = \sum a_j \bar{R}_j$  where  $\sum a_j = 0$ . For simplicity throughout this paper, we will take  $N = \sum n_j$ . Also, for  $N$  fixed we let  $\lambda_j = n_j/N$ ,  $j = 1, \dots, k$ , while the asymptotic results hold for  $n_j/N \rightarrow \lambda_j \in (0,1)$  as  $N \rightarrow \infty$ .

The following two theorems follow at once from the results in Hettmansperger (1984, Chapter 4).

Theorem 1. If  $H_0: \theta_1 = \dots = \theta_k$  is true, then  $EV=0$  and  $\text{Var } V = [N(N+1)/12] \sum a_j^2/n_j$ , where  $N = \sum n_j$ . Further, as  $N \rightarrow \infty$ ,

$$V^* = \left( \frac{12}{N+1} \right)^{1/2} \frac{V}{\left( \sum \frac{a_j^2}{\lambda_j} \right)^{1/2}} \xrightarrow{D} Z \sim n(0,1)$$

where  $Z \sim n(0,1)$  means  $Z$  has a standard normal distribution.

Theorem 2. If  $H_{AN}: \theta_j = \theta_0 + \theta c_j N^{-1/2}$ ,  $\theta > 0$ ,  $j=1, \dots, k$  is true and  $F$  has a density  $f$  with  $\int f^2(x) dx < \infty$ , then

$$V^* \xrightarrow{D} Z \sim n(\theta e, 1)$$

as  $N \rightarrow \infty$ , where

$$e = \frac{\sum a_j (c_j - \bar{c}_w)}{(\sum a_j^2 / \lambda_j)^{1/2}} \quad (12)^{1/2} \int f^2(x) dx$$



and  $\bar{c}_w = \sum \lambda_j c_j$ .

Hence, if we reject  $H_0: \theta_1 = \dots = \theta_k$  when  $V^* \geq Z_\alpha$  where  $1 - \Phi(Z_\alpha) = \alpha$  and  $\Phi(\cdot)$  is the standard normal distribution function, then for large  $N$ , the test based on  $V^*$  is approximately size  $\alpha$  and has approximate local power  $1 - \Phi(Z_\alpha - \theta e)$ . The quantity  $e$  is called the Pitman efficacy of the test. If two tests have efficacies  $e_1$  and  $e_2$ , respectively, then the efficiency of  $V_1^*$  relative to  $V_2^*$  is  $e_1^2/e_2^2$ . A test with largest  $e$  has maximum asymptotic local power and is most efficient. The best test of the form  $V^*$  is given in the next theorem.

**Theorem 3.** Given the pattern  $c_1, \dots, c_k$ ,  $e$  is maximized by  $a_j = \lambda_j(c_j - \bar{c}_w)$  where  $\lambda_j = n_j/N$ ,  $j=1, \dots, k$ , and  $\bar{c}_w = \sum \lambda_j c_j$ .

Lagrange multipliers may be used to derive the above expression for  $a_j$  as the unique solution candidate, and the Cauchy-Schwarz inequality used to confirm that  $e$  is maximized. For a proof, see the appendix.

Henceforth, we take

$$V = \sum \lambda_j (c_j - \bar{c}_w) \bar{R}_j \quad (2.1)$$

where  $\lambda_j = n_j/N$ ,  $N = \sum n_j$  and  $\bar{c}_w = \sum \lambda_j c_j$ . Furthermore, from Theorem 1, if  $H_0: \theta_1 = \dots = \theta_k$  is true,  $EV=0$ ,  $\text{Var } V = [(N+1)/12] \sum \lambda_j (c_j - \bar{c}_w)^2$  and

$$V^* = \left( \frac{12}{N+1} \right)^{1/2} \frac{\sum \lambda_j (c_j - \bar{c}_w) \bar{R}_j}{(\sum \lambda_j (c_j - \bar{c}_w)^2)^{1/2}} \xrightarrow{D} Z \sim N(0,1). \quad (2.2)$$

Hogg (1965) illustrates the substantial increase in power that results when this type of test is used rather than an omnibus test. Hogg considers the test based on sample means rather than average ranks, but the effect is similar.

It will be useful to introduce a slightly different form for  $V^*$ . Let

$$b_j = \frac{\lambda_j^{1/2} (c_j - \bar{c}_w)}{[\sum_i \lambda_i (c_i - \bar{c}_w)^2]^{1/2}}. \quad (2.3)$$

Then

$$V^* = \left( \frac{12}{N+1} \right)^{1/2} \sum b_j \lambda_j^{1/2} \bar{R}_j \quad (2.4)$$

and  $\sum \lambda_j^{1/2} b_j = 0, \quad \sum b_j^2 = 1.$

Recall, the researcher chooses a pattern  $c_1, \dots, c_k$ , or equivalently,  $b_1, \dots, b_k$ . If, on the other hand, no pattern or only a vague pattern is specified then we could use  $\text{Max } V^*$  where we maximize over a collection of choices of  $b_1, \dots, b_k$  subject to the appropriate constraints.

For example, in testing  $H_0: \theta_1 = \dots = \theta_k$  versus  $H_A: \theta_i$ 's not all equal, we would use  $\text{Max } V^*$  where the maximum is taken over all  $b_1, \dots, b_k$  such that  $\sum \lambda_j^{1/2} b_j = 0$  and  $\sum b_j^2 = 1$ . The Cauchy-Schwarz inequality implies that

$$b_j = \frac{\lambda_j^{1/2} [\bar{R}_j - \frac{N+1}{2}]}{[\sum \lambda_j (\bar{R}_j - (N+1)/2)^2]^{1/2}} \quad (2.5)$$

The constraints are satisfied and

$$\begin{aligned} \text{Max } V^* &= [(12/(n+1)) \sum \lambda_j (\bar{R}_j - (N+1)/2)^2]^{1/2} \\ &= H^{1/2} \end{aligned} \quad (2.6)$$

where  $H$  is the Kruskal-Wallis (1952) rank statistic for the one-way layout. Hence, under  $H_0$ ,  $\text{Max } V^*$  is distributed like the square root of  $\chi^2(k-1)$  and can be used to make the test.

Next, consider  $H_0: \theta_1 = \dots = \theta_k$  versus  $H_A:$

$\theta_1 \leq \dots \leq \theta_k$  with at least one strict inequality. We propose requiring  $c_1, \dots, c_k$  to be equally spaced unless there are indications to the contrary. Hence, take  $c_j = j$ ,  $j=1, \dots, k$  and the statistic (2.1) becomes

$$V = \sum \lambda_j (j - \bar{j}_w) \bar{R}_j \quad (2.7)$$

where  $\bar{j}_w = \sum j \lambda_j$ . This has a convenient asymptotic distribution theory as outlined above. When the sample sizes are equal,  $n_1 = \dots = n_k$ , we have  $V = k^{-1} \sum (j - (k+1)/2) \bar{R}_j$  and  $\text{Var } V = (k+1)(k-1)(N+1)/144$  which is quite easy to use.

If no spacing can be specified, then we would use  $\text{Max } V^*$

where the maximum is taken over choices of  $b_1, \dots, b_k$  such that  $\sum \lambda_j^{1/2} b_j = 0$ ,  $\sum b_j^2 = 1$  and  $\lambda_1^{-1/2} b_1 \leq \dots \leq \lambda_k^{-1/2} b_k$ .

Using an argument similar to that of Hogg (1965, p.1156) we have

$$\begin{aligned} \text{Max } V^* &= [(12/(N+1)) \sum \lambda_j [\hat{R}_j - (N+1)/2]^2]^{1/2} \\ &= [\bar{x}_{\text{rank}}^2]^{1/2} \end{aligned} \quad (2.8)$$

where  $\hat{R}_1 \leq \dots \leq \hat{R}_{12}$  is the isotonic regression of  $\bar{R}_1, \dots, \bar{R}_k$  with weights  $\lambda_1, \dots, \lambda_k$  and  $\bar{x}_{\text{rank}}^2$  is Chacko's (1963) rank statistic for ordered alternatives. See Barlow et.al. 1972, p. 201) for a discussion. Special tables are required for the application of  $\bar{x}_{\text{rank}}^2$  and hence when reasonable, we recommend using (2.7).

Finally, we consider the umbrella alternative for which we have  $H_0: \theta_1 = \dots = \theta_k$  versus  $H_A: \theta_1 \leq \dots \leq \theta_t \geq \theta_{t+1} \geq \dots \geq \theta_k$ . We treat the case of the peak specified first, i.e. the case when  $t$  is known. Again, unless otherwise indicated, we would take equally spaced  $c_1, \dots, c_k$ , say  $c_j = j, j=1, \dots, t$  and  $c_j = 2t-j$  for  $j=t+1, \dots, k$ , and use  $V^*$  (2.2) to carry out the test.

If the peak is unknown, we impose the restriction that  $c_1, \dots, c_k$  be equally spaced. Then there are  $k$  umbrella alternatives, including the two monotone alternatives at either extreme. We propose to use  $\text{Max } V^*$  where the maximum is taken over a finite collection of possible patterns of  $c_1, \dots, c_k$ . The index at which the maximum occurs estimates the peak. This is illustrated in section 4.

In general, there may be  $p$  possible sets of coefficients  $c_1, \dots, c_k$ . We now describe the distribution theory needed to

implement  $\max V^*$ . Let  $\tilde{V}^* = (V_1^*, \dots, V_p^*)$  be a vector of statistics with

$$V_t^* = \left( \frac{12}{N+1} \right)^{1/2} \frac{\sum_j \lambda_j (c_{tj} - \bar{c}_{tw}) \bar{R}_j}{[\sum_j \lambda_j (c_{tj} - \bar{c}_{tw})^2]^{1/2}} \quad (2.9)$$

for  $t=1, \dots, p$ . A standard calculation shows that under  $H_0$ ,

$$\text{cov}(V_s^*, V_t^*) = \frac{\sum_j \lambda_j (c_{sj} - \bar{c}_{sw})(c_{tj} - \bar{c}_{tw})}{[\sum_j \lambda_j (c_{sj} - \bar{c}_{sw})^2 \sum_j \lambda_j (c_{tj} - \bar{c}_{tw})^2]^{1/2}}. \quad (2.10)$$

In terms of  $b_{tj}$ ,  $t=1, \dots, p$ ,  $j=1, \dots, k$ , defined in (2.3) we have the covariance matrix of  $\tilde{V}^*$  given by

$$\text{cov}(\tilde{V}^*) = \tilde{B} \tilde{B}' \quad (2.11)$$

where  $\tilde{B} = (b_{tj})$ . The asymptotic distribution is given by the next theorem.

**Theorem 4.** Suppose  $H_0: \theta_1 = \dots = \theta_k$  is true. If  $N \rightarrow \infty$  in such a way that  $n_j/N \rightarrow \lambda_j \in (0,1)$ ,  $j=1, \dots, k$ , then  $\tilde{V}^* \xrightarrow{D} \tilde{B} \tilde{Z}$ , where  $\tilde{Z} \sim \text{MVN}(0, I)$ .

Standard arguments, given for example in Hettmansperger (1984, Chapter 4), show that  $\tilde{V}^*$  has a limiting  $\text{MVN}(0, \tilde{B} \tilde{B}')$  distribution, and  $\tilde{B} \tilde{Z}$ , where  $\tilde{Z} \sim \text{MVN}(0, I)$ , has the same

distribution.

It further follows that, under  $H_0$ ,

$$\max_{\sim} V^* \xrightarrow{D} \max_{\sim\sim} BZ. \quad (2.12)$$

The distribution of  $\max_{\sim\sim} BZ$  is intractable and can be computed or approximated only in a few very special cases. See, for example, Johnson and Kotz (1970). However, it is quite simple to simulate probabilities of  $\max_{\sim\sim} BZ$ . If the observed value of  $V^*$  is  $v^*$ , then the appropriate P-value of a test based on  $\max_{\sim} V^*$  is

$$P(\max_{\sim\sim} BZ \geq \max_{\sim} v^*). \quad (2.13)$$

This is easy to simulate since all we need is a vector  $Z$  of iid  $n(0,1)$  variates along with  $B$ . We illustrate this in section 4 for umbrella alternatives with unspecified peak. We also provide a table of selected simulated critical values for this problem (Table 4.1 and 4.2). Finally, note that if we store  $\bar{R}' = (\bar{R}_1, \dots, \bar{R}_k)$  and the matrix  $\Lambda = \text{diag}(\lambda_1^{1/2}, \dots, \lambda_k^{1/2})$ , then we can compute  $\max_{\sim} V^* = (12/(N+1))^{1/2} \max_{\sim\sim\sim} B\Lambda R$ .

Terpstra (1952) and Jonckheere (1954) proposed a test for ordered alternatives based on pairwise Mann-Whitney-Wilcoxon statistics. The statistic is  $J = \sum_{i < j} U_{ij}$  where  $U_{ij}$  is the number of times observations in the  $j$ th sample exceed observations in  $i$ th sample. Mack and Wolfe (1981) propose

$$A_t = \sum_{i < j \leq t} \sum U_{ij} + \sum_{t \leq i < j} \sum U_{ji} \text{ for testing an umbrella alternative}$$

with peak at  $t$ . Work of Fairley and Pearl (1984) suggests that neither the combined ranking nor pairwise ranking method is superior. Hence, for completeness, we provide statistics which are equivalent in a Pitman efficiency sense to  $V^*$  but based on pairwise rankings. This affords an easier comparison of our statistics with those of Mack and Wolfe for umbrella alternatives.

The following lemma provides the link.

Lemma. Given  $d_1, \dots, d_k$  such that  $\sum d_k = 0$  and any quantities  $Z_1, \dots, Z_k$ , we have

$$k \sum_{j=1}^k d_j Z_j = \sum_{i=1}^{k-1} \sum_{j=i+1}^k (d_j - d_i)(Z_j - Z_i).$$

The proof is by direct algebraic manipulation.

To convert  $V^*$  into a form that is in terms of pairwise comparisons, we would let  $Z_i = \bar{R}_i$  and  $d_j = \lambda_j(c_j - \bar{c}_w)$ . Then to move to pairwise rankings we replace  $\bar{R}_j - \bar{R}_i$  by  $W_{ij} = n_i^{-1} n_j^{-1} U_{ij}$ . Hence, the statistic which we consider is

$$U^* = k \sum_{i < j} \sum (d_j - d_i) W_{ij} \quad (2.14)$$

with  $d_j = \lambda_j(c_j - \bar{c}_w)$ . Under the null hypothesis

$H_0: \theta_1 = \dots = \theta_k, EU^* = 0$  and

$$\text{Var } U^* = \frac{1}{12} \left\{ \sum_{i < j} \frac{(d_j - d_i)^2}{n_i n_j} + k \sum_{i < j} \frac{d_j}{n_j} - \frac{d_i}{n_i} (d_j - d_i) \right\}. \quad (2.15)$$

See the appendix for a proof. Furthermore, under the null hypothesis,  $(\text{Var } U^*)^{-1/2} U^*$  has a limiting standard normal distribution. Because of the complexity of the variance formula,  $U^*$  is of doubtful practical value. However, if we have equal sample sizes, the formulas can be simplified considerably and  $U^*$  can be used.

Suppose  $n_1 = \dots = n_k = n$ , so  $\lambda_j = k^{-1}$  for  $j=1, \dots, k$ . Then, under the null hypothesis,  $(\text{Var } U^*)^{-1/2} U^*$  is asymptotically  $n(0,1)$  with

$$U^* = \frac{1}{n^2} \sum_{i < j} (c_j - c_i) U_{ij} \quad (2.16)$$

and, since  $\sum_{i < j} (c_j - c_i)^2 = k \sum_j (c_j - \bar{c})^2$ ,

$$\begin{aligned} \text{Var } U^* &= \frac{1}{12nk} (k+1) \sum_{j=1}^k (c_j - \bar{c})^2 \\ &= \frac{(nk+1)}{12n^2k} \sum_{j=1}^k (c_j - \bar{c})^2. \end{aligned} \quad (2.17)$$

Calculations, similar to those of Theorem 2, show that  $U^*$  and  $V^*$  have the same efficacy and same asymptotic local power. We can now illustrate the difference between our approach of selecting  $c_1, \dots, c_k$  based on equal spacings



and the Mack-Wolfe approach of piecing together two Jonckheere-Terpstra statistics for testing the umbrella alternative. For example, take  $k=5$  and  $t=3$ . Then, we have

$$n^2 U^* = U_{12} + 2U_{13} + U_{23} + U_{14} - U_{34} - 2U_{35} - U_{45} - U_{25}$$

and the Mack-Wolfe statistic is equivalent to

$$A^* = U_{12} + U_{13} + U_{23} - U_{34} - U_{35} - U_{45}.$$

The statistic  $U^*$  utilizes  $U_{14}$  and  $U_{25}$ , which make the comparisons across the peak. These are excluded from  $A^*$  and can result in some loss of efficiency. The exclusion could have unpleasant consequences in the peak unknown case, as discussed in a prototype example in section 4.

We feel that the statistic  $V^*$  (or  $U^*$  when practical) provides a natural and easy way to implement a test for a patterned alternative. If the researcher does not feel equally spaced alternatives are appropriate and can provide rough information on the spacings, then it is a simple matter to choose the appropriate weights  $c_1, \dots, c_k$ .

The approach described above extends quite easily to the additive two-way layout with  $b$  blocks and  $k$  treatments. Suppose  $X_{ijt}$ ,  $i=1, \dots, b$ ,  $j=1, \dots, k$  and  $t=1, \dots, n_{ij}$ , are mutually independent random variables with respective continuous distribution functions  $F(x-\theta_{ij})$ . Further, suppose  $\theta_{ij} = \mu + \alpha_i + \theta c_j$ ,  $\theta \geq 0$ , and  $c_1, \dots, c_k$  specify

the pattern. The additivity implies that the pattern is propagated through the  $b$  blocks. We wish to test  $H_0: \theta = 0$  versus  $H_A: \theta > 0$ .

The data should be ranked within blocks. Then  $R_{ijt}$  is the rank of  $X_{ijt}$  when ranked among the

$N_i = \sum_{j=1}^k n_{ij}$  observations in the  $i$ th block. Let  $\bar{R}_{ij}$  denote

the average rank in the  $(i,j)$  cell. Results similar to those in Theorems 1-3 show that we should use

$$V = \sum_{i=1}^b \sum_{j=1}^k \lambda_{ij}(c_j - \bar{c}_{i\cdot})\bar{R}_{ij} \quad (2.18)$$

where  $\lambda_{ij} = n_{ij}/N_i$  and  $\bar{c}_{i\cdot} = \sum_{j=1}^k \lambda_{ij}c_j$ . Under the null

hypothesis  $H_0: \theta=0$ ,  $EV=0$  and

$$\text{Var } V = \frac{1}{12} \sum_{i=1}^b (N_i+1) \sum_{j=1}^k \lambda_{ij}(c_j - \bar{c}_{i\cdot})^2. \quad (2.19)$$

Further,  $(\text{Var } V)^{-1/2} V$  has asymptotically a standard normal distribution as  $n_{ij} \rightarrow \infty$ ,  $i=1, \dots, b$ ,  $j=1, \dots, k$ , and  $b$  remains fixed or as  $b \rightarrow \infty$  and  $n_{ij}$  remains fixed. Hence, it is easy to construct approximate critical values for the test based on  $V$ .

When the sample sizes are equal,  $n_{ij}=n$  for all  $i$  and  $j$ , and the formulas can be simplified.

In this case,  $\lambda_{ij}=1/k$ ,  $N_i=nk$ ,  $\bar{c}_{i\cdot}=k^{-1} \sum_1^k c_j=\bar{c}$ , and so we can

consider:

$$\begin{aligned} V' &= \sum_{i=1}^b \sum_{j=1}^k (c_j - \bar{c}) R_{ij} \\ &= \sum_{j=1}^k (c_j - \bar{c}) R_j \end{aligned} \quad (2.20)$$

where  $R_{ij}$  is the sum of ranks in the  $(i,j)$  cell and  $R_j$  is the sum of ranks under treatment  $j$ . Further,

$$\text{Var } V' = \frac{bn^2k(nk+1)}{12} \sum_{j=1}^k (c_j - \bar{c})^2. \quad (2.21)$$

For example, if  $n=1$  and  $c_j=j$ , so we are considering the additive two-way layout with one observation per cell and ordered alternatives, then  $V'$  is Page's (1963) statistic and  $\text{Var } V' = bk^2(k^2-1)(k+1)/144$ . The statistic  $V$ , (2.18), generalizes and improves the statistics in Hettmansperger (1975).

### 3. Covariate Adjusted Tests in a One-Way Layout.

It may happen that the researcher has a set of covariate measurements available along with the treatment measurements. It then becomes necessary to correct or adjust the rank test statistic in order to test for a patterned alternative. We

will assume a parallelism model for the covariate so the same slope parameter applies to every treatment; otherwise, there is an interaction between the covariate and treatment factors, and a test for treatment differences is difficult or impossible to interpret.

It is convenient to cast the one-way layout in terms of a linear model. The observations  $X_{ij}$ ,  $j=1, \dots, k$ ,  $i=1, \dots, n_j$ , will be arranged in a vector  $\tilde{Y}$  of length  $N=\sum n_j$ . The first  $n_1$  components comprise the first sample and so on. Then, if there are  $q$  covariates, the model is written as:

$$\tilde{Y} = \tilde{1}\alpha + \tilde{W}\tilde{\theta} + \tilde{Z}\tilde{\gamma} + \tilde{e},$$

where  $\tilde{1}$  is an  $N \times 1$  vector of ones,  $\tilde{W}$  is an  $N \times k$  matrix in which the  $i$ th column has ones corresponding to the  $i$ th sample and zeros otherwise,  $\tilde{Z}$  is an  $N \times q$  matrix of covariates,  $\alpha$  is the scalar intercept parameter,  $\tilde{\theta}' = (\theta_1, \dots, \theta_k)$  are the treatment parameters of interest and  $\tilde{\gamma}' = (\gamma_1, \dots, \gamma_q)$  gives the slope parameters. The mean centered matrices will be denoted by  $\tilde{W}_c$  and  $\tilde{Z}_c$ . For example,  $\tilde{W}_c = \tilde{W} - N^{-1}\tilde{1}\tilde{1}'\tilde{W}$ . The matrix  $\tilde{W}_c'\tilde{W}_c$  is singular and we will assume that the matrix  $\tilde{Z}_c'\tilde{Z}_c$  is nonsingular. Finally, the vector  $\tilde{e}' = (e_1, \dots, e_N)$  of random errors are assumed to have independent and identically distributed random variable components from a continuous distribution with unique median 0.

As before, we are interested in testing  $H_0$  :

$$\theta_1 = \dots = \theta_k \text{ versus } H_A: \theta_j = \theta_0 + \theta c_j, j=1, \dots, k,$$

$\theta > 0$  and  $c_1, \dots, c_k$  a given pattern. The test will be based on average ranks, after the data have been adjusted for the covariates. Suppose  $\hat{\gamma}$  is an estimate of  $\gamma$  based on the reduced model under  $H_0$ ,  $Y = 1\alpha + Z\gamma + e$ . We then rank  $Y - Z\hat{\gamma}$  and use  $\bar{R}_1, \dots, \bar{R}_k$ , the average ranks of the  $k$  adjusted samples, in equation (2.1) to form a test statistic. The following theorem, which is proven in the Appendix, provides the asymptotic result needed to approximate critical values.

**Theorem 5.** Suppose  $H_0: \theta_1 = \dots = \theta_k$  is true. If  $N^{1/2}(\hat{\gamma} - \gamma)$  is bounded in probability and if  $N^{-1} Z_c' Z_c$  has a positive definite limit as  $N \rightarrow \infty$ , then

$$V^* = \left( \frac{12}{N+1} \right)^{1/2} \frac{1}{\sigma} \sum_{j=1}^k \lambda_j (c_j - \bar{c}_w) \bar{R}_j \xrightarrow{D} Z \sim n(0,1) \quad (3.1)$$

where

$$\sigma^2 = \sum_{j=1}^k \lambda_j (c_j - \bar{c}_w)^2 - N^{-1} \tilde{c}' W_c' Z_c (Z_c' Z_c)^{-1} Z_c' W_c \tilde{c} \quad (3.2)$$

$$\tilde{c}' = (c_1 - \bar{c}_w, \dots, c_k - \bar{c}_w) .$$

This result corresponds to (2.2) and provides the covariate adjustment for the standardization of the test statistic. Hence, we cannot simply adjust the data and apply (2.2); the variance must be adjusted also.

Note that the variance adjustment, given as the second term in (3.2), is non-negative. This follows since  $Z_c(Z_c'Z_c)^{-1}Z_c'$  is a non-negative definite projection matrix. When there is only one covariate, the correction term in (3.2) is  $[\sum \lambda_j (c_j - \bar{c}_w) \bar{z}_{cj}]^2 / [N^{-1} \sum (z_i - \bar{z})^2]$  where  $\bar{z}_{cj}$  is the average of the  $n_j$  components of  $Z_c$ , corresponding to the  $j$ th sample. This means that if (2.2) rather than (3.1) is applied to the adjusted data, an asymptotically conservative test will result. It is easy to construct examples for which the correction term is substantial.

The result in Theorem 5 is quite general and can be applied to an additive AOV model in which the other factors are identified as the covariates. When there are an equal number of observations per cell, balanced data, the correction term in (3.2) is 0, since  $W_c'Z_c = 0$ . Then the simpler result in (2.2) can be applied directly to the average ranks of the adjusted data.

#### 4. Examples and Discussion

As an example, we consider data from an age discrimination suit filed in a United States District Court in South Carolina in 1980. Because this case was settled out of court, the names of individuals and the company involved will not be mentioned. We say only that related to a company reduction in force, the company terminated a number of employees based on evaluation of job performance, that those terminated were near retirement age, and that those individuals brought suit.

In one aspect of the case, twelve individuals with the same job titles were ranked from one to twelve as to their job performance, with "one" representing the least proficient and "twelve" the most proficient. No measure of job proficiency other than the relative rankings was provided by the company. Month and year of birth data are given in Table 4.1. The company claimed that age would have no statistical bearing on job performance.

Table 4.1 may be viewed as giving rank summaries of observations of a continuous random variable, performance, with twelve independent samples of size one, the samples being indexed by birthdate. As the alternative hypothesis, we take

$H_A: \theta_1 \leq \theta_2 \leq \dots \leq \theta_t \geq \theta_{t+1} \geq \theta_{t+2} \geq \dots \geq \theta_{12}$ , with at least one strict inequality and  $t$  unknown. The unknown peak perspective reflects one item that was of interest in the case. Namely, whether proficiency on this job tends to increase with age up to some point, then tends to decrease with age.

Month and Year of Birth	Rank of Birth Date	Performance Rank
11 - 1915	12	1
2 - 1917	11	3
12 - 1918	10	4
4 - 1919	9	7
9 - 1921	8	11
11 - 1921	7	6
12 - 1926	6	10
2 - 1927	5	12
6 - 1929	4	9
9 1930	3	8
9 1931	2	5
10 1931	1	2

Table 4.1

We use an analysis for general  $k$  and equal sample sizes. The test statistic is to be sensitive to any of the following equally spaced umbrella patterns  $c_1, \dots, c_k$ ,

where  $h > 0$ :

$$\begin{array}{ll}
 \theta_0, \theta_0 - h, \theta_0 - 2h, \dots, \theta_0 - (k-1)h & \text{peak at } c_1 \\
 \theta_0, \theta_0 + h, \theta_0, \theta_0 - h, \dots, \theta_0 - (k-3)h & \text{peak at } c_2 \\
 \theta_0, \theta_0 + h, \theta_0 + 2h, \theta_0 + h, \dots, \theta_0 - (k-5)h & \text{peak at } c_3 \\
 & (4.1) \\
 \theta_0, \theta_0 + h, \theta_0 + 2h, \dots, \theta_0 + (k-1)h & \text{peak at } c_k
 \end{array}$$



We use as the test statistic  $V_{\max}^* \equiv \max\{V_1^*, \dots, V_k^*\}$ , where  $V_t^*$  is given in (2.9). When  $\lambda_1 = \lambda_2 = \dots = \lambda_k = 1/k$ , then  $\bar{c}_{tw} = \bar{c}_t$  and we have

$$V_t^* = \left[ \frac{12}{k(N+1)} \right]^{1/2} \sum_j \frac{c_{tj} - \bar{c}_t}{[\sum_r (c_{tr} - \bar{c}_t)^2]^{1/2}} \bar{R}_j. \quad (4.2)$$

Since for each  $t$  the coefficients of the  $\bar{R}_j$ 's are centered and normed, these coefficients do not depend on  $\theta_0$  or  $h$ . Hence, for  $t=1, 2, \dots, k$ , we may take  $c_{tj}=j$  for  $j=1, 2, \dots, t$  and  $c_{tj}=2t-j$  for  $j=t+1, \dots, k$ . Table 4.2 provides .05 and .01 critical values for  $V_{\max}^*$  when  $n_1=n_2=\dots=n_k=1$ .

$\alpha \backslash k$	5	6	7	8	9	10	11	12	15	20	25	30
.05	1.94	2.02	2.10	2.14	2.17	2.19*	2.22	2.21	2.26	2.27	2.29	2.30
.01	-	2.19	2.31	2.40	2.48	2.53*	2.59	2.62	2.69	2.75	2.81	2.82

Table 4.2 Critical values of the null distribution of  $V_{\max}^*$  when  $n_1=n_2=\dots=n_k=1$ . Entries through  $k=9$  are exact, while entries thereafter are based on simulations.

When the common sample size is one and  $H_0$  is true,  $\bar{R}_1, \bar{R}_2, \dots, \bar{R}_k$  is a permutation of the integers  $1, 2, \dots, k$ , with all  $k!$  permutations being equally likely. One may generate on a computer all the permutations by taking as a seed permutation  $12\dots k$ , and use  $k-1$  nested loops as follows. Let the  $i$ th innermost loop,  $i=1, 2, \dots, k-1$ , cycle the first

$i+1$  positions of the current permutation to obtain the next permutation. Say, old position 1  $\rightarrow$  new position 2, old position 2  $\rightarrow$  new position 3, ..., old  $(i+1)$ st position  $\rightarrow$  new position 1. Let this loop act  $i+1$  times, each time generating the next permutation. For each new permutation the value of  $V_{\max}^*$  is computed. In this way the exact distribution of  $V_{\max}^*$  will be obtained. One also may obtain approximate critical values for large  $k$  by using random permutations of the integers  $1, 2, \dots, k$  and empirical distributions.

For example, when  $k=5$  the maximum value of  $V_{\max}^*$  is exactly 2, occurring when  $\bar{R}_1 \bar{R}_2 \dots \bar{R}_5 = 12345$  or  $54321$ . The next largest value is 1.9414, which occurs for the permutations 12354, 12453, 35421, and 45321. Let  $W$  denote the range of  $V_{\max}^*$  after values are truncated to two decimal places. Then  $1.94 = \min \{w \in W: P_0(V_{\max}^* \geq w) \leq .05\}$ . Hence in the table, 1.94 is the .05 critical value ( $6/5! = .05$ ). There is no .01 critical value. Each simulation entry in the table is based on at least 40,000 random permutations. As an explanation to the decrease from 2.22 to 2.21 in the row of .05 critical values, we note that  $V_{\max}^*$  is discrete with gaps in its range and some values of  $W$  are much more likely to occur than their neighbors. For instance, when

$k=11$ ,  $2.20 \notin W$  while  $P_0(V_{\max}^*=2.21) \approx .003$ ,  $P_0(V_{\max}^*=2.22) \approx .0007$ , based on 120,000 permutations.

For the job performance data,  $\bar{R}_1=2$ ,  $\bar{R}_2=5, \dots, \bar{R}_{12}=1$  and  $v_{\max}^*=2.8795 \geq 2.62$ , so that  $H_0$  is rejected at the .01 significance level. A simulation approximation of the p-value is .0015.

Were the asymptotic setting of Theorem 4 appropriate,  $V_{\max}^*$ , with  $V_t^*$  as in (4.2), could be compared to the entries in Table 4.3. These are critical values of the maximum component of  $BZ$ , where  $Z \sim MVN(0, I)$  and  $b_{tj} =$

$(c_{tj} - \bar{c}_t)[\sum_r (c_{tr} - \bar{c}_t)^2]^{-1/2}$ , for  $j=1, 2, \dots, k$  and  $t=1, 2, \dots, k$ , with  $c_{tj}=j$  for  $j=1, 2, \dots, t$  and  $c_{tj}=2t-j$  for  $j=t+1, \dots, k$ .

Except for the entries when  $k=2$ , the critical values in Table 4.3 were obtained by simulation using empirical cumulative distributions. When  $k=2$  the two umbrella patterns are  $c_{11}=1$ ,  $c_{12}=0$  and  $c_{21}=1$ ,  $c_{22}=2$ . Hence we seek critical values for  $\max\{-2^{-1/2}(Z_1 - Z_2), 2^{-1/2}(Z_1 - Z_2)\} = 2^{-1/2}|Z_1 - Z_2|$ , where  $Z_1$  and  $Z_2$  are iid standard normal. Hence the critical values would be identical to those for  $|Z_1|$ .

$\alpha \backslash k$	2	3	4	5	6	7	8	9	10	15	20	30
.05	1.96*	2.12	2.19	2.24	2.26	2.27	2.28	2.29	2.30	2.31	2.31	2.32
.01	2.58*	2.73	2.78	2.83	2.85	2.86	2.87	2.89	2.89	2.91	2.91	2.91

Table 4.3 Approximate (except for  $k=2$ ) critical values for the limiting null distributions of  $V_{\max}^*$  when  $\lambda_1=\lambda_2=\dots=\lambda_k=1/k$  and  $V_t^*$  is given in (4.2).

We now compare the unknown peak statistic  $A_{\hat{g}}$  as presented by Mack and Wolfe (1981) to  $Z_{\max}^*$  on an example which shows that the former may be dramatically affected by minor perturbations in the data. We begin with their statistic

$$A_{\hat{g}} = \sum_{1 \leq i < j \leq \ell} U_{ij} + \sum_{\ell \leq i < j \leq k} U_{ji}. \quad (4.3)$$

used when the umbrella peak is known to occur a priori at

group  $\ell$ . Letting  $N_1 = \sum_{i=1}^{\ell} n_i$ ,  $N_2 = \sum_{i=\ell}^k n_i$  and  $N = \sum_{i=1}^k n_i$ ,

then the null asymptotic distribution of  $[A_{\hat{g}} - \mu_0(A_{\hat{g}})]/\sigma_0(A_{\hat{g}})$  is standard normal with

$$\mu_0(A_{\hat{g}}) = 4^{-1}[N_1^2 + N_2^2 - \sum_{i=1}^k n_i^2 - n_{\ell}^2]$$

and

$$\begin{aligned} \sigma_0^2(A_{\hat{g}}) = & 72^{-1}[2(N_1^3 + N_2^3) + 3(N_1^2 + N_2^2) - \sum_{i=1}^k n_i^2(2n_i + 3) - n_{\ell}(2n_{\ell} + 3) \\ & + 12n_{\ell}N_1N_2 - 12n_{\ell}^2N]. \end{aligned}$$

In the unknown peak case the statistic is

$$\hat{A}_0 = \sum_{t=1}^k x_t (A_t - \mu_0(A_t)) / \sigma_0(A_t) \quad (4.4)$$

where  $x_t = 0$  if group  $t$  is not estimated as being a peak group, and  $x_t = 1/r$ , otherwise, with  $r$  being the number of groups estimated as peak groups. To estimate which are the peak groups one computes

$$B_t = \sum_{\substack{i=1 \\ i \neq t}}^k U_{it},$$

then  $B_t^* = [B_t - \mu_0(B_t)] / \sigma_0(B_t)$ , for  $t=1, 2, \dots, k$ , with  $\mu_0(B_t) = n_t(N - n_t)/2$  and  $\sigma_0^2(B_t) = n_t(N - n_t)(N + 1)/12$ . Group  $t$  is estimated as a peak group if  $B_t^* = \max\{B_1^*, \dots, B_k^*\}$ .

As seen in (4.3) Mann-Whitney counts are not computed for groups on opposite sides of the peak. Rather than looking at the overall pattern, a sample pattern is viewed in two pieces, each piece contributing circumstantial evidence or not to one of the two double sums in (4.3). The absence of across the peak comparisons can present a problem, as we now see.

Consider four samples of size 10 with ranks as in Table 4.4. Group two is estimated as the peak group ( $B_2 = 226$ ,  $B_4 = 225$ ).

with  $A_2=251$ ,  $\mu_0(A_2)=200$  and  $\sigma_0(A_2)=35.05$ . This gives  $A_2^*=1.46$  and a p-value satisfying  $p \gg .1$ , by Table 1 of Mack and Wolfe.

Samples 1 and 2 conform perfectly to the leftmost monotone increasing pattern of an umbrella, whereas samples 2, 3, and 4 conform hardly at all to a rightmost monotone decreasing pattern of an umbrella. Based on  $A_2^*$ , the sum of two Jonckheere's statistics, we fail to reject  $H_0$  at any reasonable significance level.

Sample	1	2	3	4
Ranks	10	39	33	40
	9	38	30	37
	8	35	27	36
	7	34	24	32
	6	29	21	31
	5	28	18	26
	4	23	15	25
	3	22	13	20
	2	17	12	19
	1	16	11	14

Table 4.4

Suppose that the observations ranked, say, 37th and 38th were misranked, and should be switched. Now group four is estimated as the peak group ( $B_3=225$ ,  $B_4=226$ ) and  $A_4=450$ , with  $\mu_0(A_4)=300$  and  $\sigma_0(A_4)=41.4$ . Hence  $A_4^*=3.62$ , highly significant, with a p-value satisfying  $p \ll .01$ . When Mann-Whitney counts from all possible pairs of groups are

brought to bear, there is great support for an overall monotone increasing trend in group locations. Thus the failure to compare groups across the peak can be problematic.

To use  $V_{\max}^*$  and the data in Table 4.4 we need  $\bar{R}_1=5.5$ ,  $\bar{R}_2=28.1$ ,  $\bar{R}_3=20.4$ , and  $\bar{R}_4=28.0$ . This gives  $V_{\max}^*=3.617$ . With the 37 and 38 exchanged,  $V_{\max}^*=3.641$ . Either case is highly significant, according to Table 4.3.

## APPENDIX

### Proof of Theorem 3

We find the values of  $a_1, a_2, \dots, a_k$  which maximize

$$(\Sigma a_j c_j) / (\Sigma a_j^2 / \lambda_j)^{1/2} \quad (1)$$

subject to  $\Sigma a_j = 0$ , where  $c_1, \dots, c_k$  and  $\lambda_1, \dots, \lambda_k$  are known.

Define  $f(a_1, \dots, a_k, \lambda) = (\Sigma a_j c_j) / (\Sigma a_j^2 / \lambda_j)^{1/2} + \lambda \Sigma a_j$ .

Then  $\partial f / \partial a_i = A^{-2} \{ A c_i - (\Sigma a_j c_j) a_i \lambda_i^{-1} A^{-1} \} + \lambda$ , where

$A = (\Sigma a_j^2 / \lambda_j)^{1/2}$ . Set  $\partial f / \partial a_i = 0$  for  $i = 1, 2, \dots, k$  and use the fact that  $\Sigma a_j = 0$ . First,

$$\{ A^2 c_i \lambda_i - (\Sigma a_j c_j) a_i \} / A^3 + \lambda \lambda_i = 0 \quad (2)$$

for all  $i$ . Sum the  $k$  equations in (2) to get  $A^2 (\Sigma c_j \lambda_j) =$

$-\lambda A^3 \Sigma \lambda_j$ , or  $\lambda = -(\Sigma c_j \lambda_j) / A$ . Substituting in (2) and

solving for  $a_i$  yields  $a_i = \lambda_i A^2 (c_i - \Sigma \lambda_j c_j) / \Sigma a_j c_j$ .

When  $\tilde{a}^*$  is a critical vector of expression (1), so is any scalar multiple of  $\tilde{a}^*$ . Hence we may reduce our considerations to the critical vectors  $\tilde{a}^*$  and  $-\tilde{a}^*$ , where  $a_i^* = \lambda_i (c_i - \bar{c}_w)$  with  $\bar{c}_w = \Sigma \lambda_j c_j$ . Notice that the vectors  $\tilde{a}^*$  and  $-\tilde{a}^*$ , when substituted into expression (1), yield results of opposite sign, and these are the only critical vectors, up to scalar multiples.

To see that  $\tilde{a} = \tilde{a}^*$  maximizes (1), we use the Cauchy-Schwarz inequality. Now

$$\begin{aligned} (\Sigma a_i c_i) / (\Sigma a_i^2 / \lambda_i)^{1/2} &= \{ \Sigma (a_i \lambda_i^{-1/2}) \lambda_i^{1/2} (c_i - \bar{c}_w) \} / (\Sigma a_i^2 / \lambda_i)^{1/2} \\ &\leq \{ \Sigma \lambda_i (c_i - \bar{c}_w)^2 \}^{1/2}, \end{aligned}$$



and algebra shows that when  $\tilde{a} = \tilde{a}^*$ , expression (1) equals the upper bound. Hence  $\tilde{a}^*$  is the unique solution which maximizes expression (1).

Proof of (2.15). Let  $V_{ij} = (d_j - d_i)U_{ij}/n_i n_j$ .

$$\text{Var}(\sum_{i < j} \sum V_{ij}) = \sum_{i < j} \sum \text{Var}(V_{ij}) + \text{Covariance terms.}$$

$$\begin{aligned} \text{Covariance terms} &= \sum_{i < j} \sum \left[ \sum_{\substack{s=1 \\ s \neq i}}^{j-1} \text{Cov}(V_{ij}, V_{sj}) + \sum_{\substack{t=1+1 \\ t \neq j}}^k \text{Cov}(V_{ij}, V_{it}) \right. \\ &\quad \left. + \sum_{\substack{s=1 \\ s \neq j}}^{i-1} \text{Cov}(V_{ij}, V_{si}) + \sum_{\substack{t=j+1 \\ t \neq i}}^k \text{Cov}(V_{ij}, V_{jt}) \right] \\ &= \sum_{i < j} \sum_{s=1}^{j-1} (d_j - d_i)(d_j - d_s) \left( \frac{1}{12n_j} \right) + \\ &\quad \sum_{t=1+1}^k (d_j - d_i)(d_t - d_i) \left( \frac{1}{12n_i} \right) + \sum_{s=1}^{i-1} (d_j - d_i)(d_i - d_s) \left( \frac{-1}{12n_i} \right) \\ &\quad + \sum_{t=j+1}^k (d_j - d_i)(d_t - d_j) \left( \frac{-1}{12n_j} \right) \Bigg] , \end{aligned}$$

where the covariances are given in Tryon and Hettmansperger (1973, Section 4). Now

$$\begin{aligned} \text{covariance terms} &= \frac{1}{12} \sum_{i < j} \sum (d_j - d_i) \left[ \sum_{\substack{s=1 \\ s \neq i, j}}^k (d_j - d_s) \frac{1}{n_j} + \sum_{\substack{t=1 \\ t \neq i, j}}^k (d_t - d_i) \left( \frac{1}{n_i} \right) \right] \\ &= \frac{1}{12} \sum_{i < j} \sum (d_j - d_i) \left\{ \frac{1}{n_j} [(k-2)d_j + d_i + d_j] - \frac{1}{n_i} [(k-2)d_i + d_i + d_j] \right\} \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{12} \sum_{i < j} \sum (d_j - d_i) \left[ k \left( \frac{d_j}{n_j} - \frac{d_i}{n_i} \right) \right. \\
&\quad \left. - (d_j - d_i) \left( \frac{1}{n_i} + \frac{1}{n_j} \right) \right] \\
&= \frac{k}{12} \sum_{i < j} \sum (d_j - d_i) \left( \frac{d_j}{n_j} - \frac{d_i}{n_i} \right) - \frac{1}{12} \sum_{i < j} \sum (d_j - d_i)^2 \left( \frac{1}{n_i} + \frac{1}{n_j} \right).
\end{aligned}$$

Further,  $\text{Var } v_{ij} = (d_j - d_i)^2 (n_i + n_j + 1) / (12 n_i n_j)$  and (2.15) follows by combining the variance and covariance terms.

#### Proof of Theorem 5.

Let  $W^*$  be the matrix formed by dropping the first column of  $W$ . Consider the model  $\underline{Y} = \underline{1}\alpha + W^*\underline{\theta}^* + Z\underline{\gamma} + \underline{e}$  where  $\underline{\theta}^{*'} = (\theta_2, \dots, \theta_k)$ . Now the design  $X = [W^*, Z]$  is such that  $N^{-1}X_C'X_C$  has a positive definite limit.

Let

$$\frac{1}{N^{1/2}} S_j = \left( \frac{12}{N+1} \right)^{1/2} \lambda_j \left( \bar{R}_j - \frac{N+1}{2} \right)$$

for  $j=2, \dots, k$ , where  $\bar{R}_j$  is based on  $\underline{Y} - Z \hat{\underline{\gamma}}$ .

Let  $N^{-1/2} S_2^*$  be the  $(k-1) \times 1$  vector of such components.

Now, when  $N^{1/2}(\hat{\underline{\gamma}} - \underline{\gamma})$  is bounded in probability under  $H_0$

for large  $N$ , the result in the proof of Theorem 5.3.2, Hettmansperger (1984), shows that

$$\frac{1}{N^{1/2}} \underline{S_2^*} \xrightarrow{D} \underline{Z} \sim \text{MVN}(0, \Lambda^*)$$

where  $\Lambda^* = \lim N^{-1} [W_C^{*'} W_C^* - W_C^{*'} Z_C (Z_C' Z_C)^{-1} Z_C' W_C^*]$ .

Now define

$$\underline{S_2'} = (\underline{S_1}, \underline{S_2^{*'}}).$$

Since,  $\underline{S_1} = - \sum_{j=2}^k \underline{S_j}$ , we have

$$N^{-1/2} \underline{S_2} = \begin{pmatrix} -\underline{1'} \\ \underline{I} \end{pmatrix} N^{-1/2} \underline{S_2^*} \xrightarrow{D} \underline{Z} \sim \text{MVN}(0, \Lambda)$$

where  $\Lambda = \lim N^{-1} [W_C' W_C - W_C' Z_C (Z_C' Z_C)^{-1} Z_C' W_C]$ .

Now,

$$\begin{aligned} \left( \frac{12}{N+1} \right)^{1/2} \sum_{j=1}^k \lambda_j (c_j - \bar{c}_W) \bar{R}_j &= \left( \frac{12}{N+1} \right)^{1/2} \sum_{j=1}^k \lambda_j (c_j - \bar{c}_W) \left( \bar{R}_j - \frac{N+1}{2} \right) \\ &= N^{-1/2} \sum_{j=1}^k (c_j - \bar{c}_W) S_j \end{aligned}$$

which converges in distribution to a normal random variable with mean 0 and variance  $\underline{c'} \Lambda \underline{c}$  where  $\underline{c'} = (c_1 - \bar{c}_W, \dots, c_k - \bar{c}_W)$ .

Finally, note that  $N^{-1} W_C' W_C$  has  $i$ th diagonal element  $\lambda_i(1 - \lambda_i)$  and off diagonal elements  $-\lambda_i \lambda_j$ . Straightforward calculation shows  $\sum \lambda_j (c_j - \bar{c}_W)^2 = \underline{c'} (N^{-1} W_C' W_C) \underline{c}$ , and the theorem is proved.

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